Assignment # 8	Due on December 25th in class

1. Given the Banach space $(C[0,1], \|\cdot\|)$ where $\|\cdot\|$ is defined as

$$\|x\| = \max_{t\in[0,1]} |x(t)| \ ,$$

prove that this Banach space cannot be an inner product space (with the above defined norm induced by the innder product as $\|x\| = \langle x, x\rangle)$.

Hint: Just need to show that the parallelogram equality/identity does not hold for the above defined norm of $C[0,1]$. In other words, just need to find $x, y \in C[0,1]$, such that

$$\|x+y\|^2 + \|x-y\|^2 \neq 2(\|x\|^2 + \|y\|^2) \ .$$

2. If an inner product space X is real (in other words, X is over $\mathbb{R}$), prove that

$$\|x\| = \|y\| \quad \text{if and only if} \quad \langle x+y, x-y\rangle = 0 \ .$$

What does that mean if $X = \mathbb{R}^n$ with $n \geq 2$? In other words, try giving a geometric explanation of the fact you just proved for the case $X = \mathbb{R}^n$ with $n \geq 2$.

Solution:

1.

Proof. We just need to check against the parallelogram identity to show that it does not hold. Just choose

$$x(t) = t \quad \text{and} \quad y(t) = 1 - t.$$

We can check that $\|x\| = 1$, $\|y\| = 1$, $\|x + y\| = 1$ and $\|x - y\| = 1$, which ensures that the parallelogram does not hold.

□

2.

Proof. Just note that $\|x\| = \|y\|$ is equivalent to $\langle x, x\rangle = \langle y, y\rangle$. As

$$\begin{aligned}
\langle x + y, x - y\rangle &= \langle x, x\rangle - \langle x, y\rangle + \langle y, x\rangle - \langle y, y\rangle \\
&= \langle x, x\rangle - \langle x, y\rangle + \overline{\langle x, y\rangle} - \langle y, y\rangle \\
&= \langle x, x\rangle - \langle x, y\rangle + \langle x, y\rangle - \langle y, y\rangle \\
&= \langle x, x\rangle - \langle y, y\rangle,
\end{aligned}$$

we are done.

□